

GREEN'S THEOREM FOR CROSSED PRODUCTS BY HILBERT C^* -BIMODULES

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ABSTRACT. Green's theorem gives a Morita equivalence $C_0(G/H, A) \rtimes G \sim A \rtimes H$ for a closed subgroup H of a locally compact group G acting on a C^* -algebra A . We prove an analogue of Green's theorem in the case $G = \mathbb{Z}$, where the automorphism generating the action is replaced by a Hilbert C^* -bimodule.

1. INTRODUCTION

The crossed product $A \rtimes X$ of a C^* -algebra A by a Hilbert $A - A$ bimodule X , as defined in [2], is a generalization of the crossed product $A \rtimes_{\alpha} \mathbb{Z}$ of A by an automorphism α of A . Given an automorphism α of A one can twist the trivial bimodule ${}_A A_A$ replacing the right structure by defining $x \cdot_{\alpha} a = x\alpha(a)$ and $\langle x, y \rangle_R^{\alpha} = \alpha^{-1}(a^*b)$ for $a, x, y \in A$, to get a C^* -bimodule, denoted by A_{α} , satisfying $A \rtimes_{\alpha} \mathbb{Z} \cong A \rtimes A_{\alpha}$ canonically.

Green's theorem, as stated in [5, Theorem 4.22], gives a Morita equivalence $C_0(G/H, A) \rtimes G \sim A \rtimes_{\alpha|_H} H$ for a general locally compact C^* -dynamical system (A, G, α) and a closed subgroup $H \leq G$. In the special case $G = \mathbb{Z}$, $H = n\mathbb{Z}$, for $n \in \mathbb{N}$, we have $G/H = \mathbb{Z}_n$ so that $C_0(G/H, A) = C_0(\mathbb{Z}_n, A) \cong A^n$ (n -fold direct sum) and $(A, H, \alpha|_H) = (A, n\mathbb{Z}, \alpha|_{n\mathbb{Z}}) \cong (A, \mathbb{Z}, \alpha^n)$ so that $A \rtimes_{\alpha|_H} H \cong A \rtimes_{\alpha^n} \mathbb{Z}$, where α also denotes the single automorphism generating the action of \mathbb{Z} on A , and α^n its n -th composition power. Then, for this special case, we have the Morita equivalence $A^n \rtimes_{\sigma} \mathbb{Z} \sim A \rtimes_{\alpha^n} \mathbb{Z}$ for a certain action σ on A^n . Translating this into the C^* -bimodule language we get $A^n \rtimes A_{\sigma}^n \sim A \rtimes A_{\alpha^n} \cong A \rtimes [A_{\alpha}]^{\otimes n}$, where we use the isomorphism $A_{\alpha^n} \cong [A_{\alpha}]^{\otimes n}$ (n -fold tensor product).

In this context, we show that one can replace A_{α} by a general right full Hilbert $A - A$ bimodule X and establish a Morita equivalence of the form

$$A^n \rtimes X_{\sigma}^n \sim A \rtimes X^{\otimes n}.$$

We obtain this as a consequence of Theorem 3.1, which states a Morita equivalence of the form

$$(A_1 \oplus \cdots \oplus A_n) \rtimes (X_1 \oplus \cdots \oplus X_n)_{\sigma} \sim A_1 \rtimes (X_1 \otimes \cdots \otimes X_n),$$

for a “cycle” of bimodules ${}_A X_1 X_2, {}_{A_2} X_2 X_3, \dots, {}_{A_{n-1}} X_{n-1} X_n, {}_{A_n} X_n X_1$, the especial case $A_i = A$, $X_i = X$, $i = 1, \dots, n$, giving the desired result.

Date: Received: xxxxxx; Revised: yyyyyy; Accepted: zzzzzz.

Partially supported by Proyecto Fondo Clemente Estable FCE2007.731.

2010 *Mathematics Subject Classification.* Primary 46L08; Secondary 46L55, 46L05.

Key words and phrases. Green's theorem, Morita equivalence, crossed product, Hilbert module.

2. PRELIMINARIES

2.1. C^* -modules, C^* -bimodules, equivalence bimodules and fullness. A *right Hilbert B -module* X_B is defined as a vector space X equipped with a right action of the C^* -algebra B and a B -valued right inner product, which is complete with respect to the induced norm. A left Hilbert A -module ${}_A X$ is defined analogously. A *Hilbert $A - B$ bimodule* ${}_A X_B$ is a vector space X with left and right compatible Hilbert C^* -module structures over C^* -algebras A and B , respectively. Compatibility means that $\langle x, y \rangle_L \cdot z = x \cdot \langle y, z \rangle_R$, for all $x, y, z \in X$. We say that a Hilbert $A - B$ bimodule is *right full* if $\langle X, X \rangle_R = B$, where $\langle X, X \rangle_R = \overline{\text{span}}(\{\langle x, y \rangle_R : x, y \in X\})$, $\overline{\text{span}}$ denoting the closed linear spanned set. Left fullness is defined analogously. Finally, an *equivalence bimodule* is a Hilbert $A - B$ bimodule ${}_A X_B$ which is right full and left full. When an equivalence bimodule ${}_A X_B$ exists the C^* -algebras A and B are said to be *Morita equivalent*, a situation denoted $A \sim B$. See [3] for reference.

2.2. Operations with subspaces. For linear subspaces X, X_1, \dots, X_n of a fixed normed $*$ -algebra C we define

$$\begin{aligned} \sum X_i &\equiv X_1 + X_2 + \dots + X_n \equiv \overline{\{x_1 + x_2 + \dots + x_n : x_i \in X_i\}}, \\ \prod X_i &\equiv X_1 X_2 \dots X_n \equiv \overline{\{\sum_k x_{1k} x_{2k} \dots x_{nk} : x_{ik} \in X_i\}}, \quad X^* \equiv \{x^* : x \in X\}. \end{aligned}$$

If Y_1, \dots, Y_n is another family of subspaces and $\overline{X_i} = \overline{Y_i}$ for $i = 1, \dots, n$ then $\sum X_i = \sum Y_i$ and $\prod X_i = \prod Y_i$. Consequently, equalities of the form $XY = \overline{X}Y$, $X + Y = \overline{X} + Y$, etc. hold for subspaces X and Y . Also, the following properties are easily checked for subspaces X, Y, Z .

1. $(X + Y) + Z = X + Y + Z = X + (Y + Z)$, 2. $X + Y = Y + X$,
3. $(XY)Z = XYZ = X(YZ)$, 4. $X(Y + Z) = XY + XZ$,
5. $(X + Y)^* = X^* + Y^*$, 6. $(XY)^* = Y^* X^*$, 7. $(X^*)^* = X$.

For a general family of subspaces $\{X_i\}_{i \in I}$ we extend the definition of sum as

$$\sum X_i \equiv \overline{\{\sum_{i \in I_0} x_i : I_0 \subseteq I \text{ finite}, x_i \in X_i\}}.$$

For every such a family and a subspace X we have

$$8. X(\sum X_i) = \sum X X_i, \quad 9. (\sum X_i)^* = \sum X_i^*.$$

2.2.1. Let C be a fixed normed $*$ -algebra, $A \subseteq C$ a $*$ -subalgebra and $X \subseteq C$ a linear subspace such that

1. $AX \subseteq X$, 2. $XA \subseteq X$, 3. $X^* X \subseteq A$, 4. $XX^* \subseteq A$.

For $k \in \mathbb{Z}$, we define $X^k = XX \dots X$ (k times) if $k \geq 1$, $X^0 = A$ and $X^k = (X^*)^{-k}$ if $k \leq -1$. We have $X^k X^l \subseteq X^{k+l}$ for all $k, l \in \mathbb{Z}$, and $X^k X^l = X^{k+l}$ if $kl > 0$. Denote with $A[X]$ the closed $*$ -subalgebra of C generated by $A \cup X$. That is

$$A[X] = C^*(A \cup X) = \sum_{k \in \mathbb{Z}} X^k.$$

2.2.2. With $A, X \subseteq C$ as before, let $B \subseteq C$ be a $*$ -subalgebra. Note that

$$\text{if } BA = AB \text{ and } BX = XB \text{ then } BA[X] = A[X]B.$$

Indeed, in this case $BX^k = X^k B$ for all $k \in \mathbb{Z}$ and then

$$BA[X] = B \sum_k X^k = \sum_k BX^k = \sum_k X^k B = A[X]B.$$

In a similar fashion, we can prove that

$$\text{if } BA = A \text{ and } BX = X \text{ then } BA[X] = A[X].$$

2.2.3. If in the context of 2.2.1 C is a C^* -algebra and A and X are closed, then A is a C^* -algebra and X a Hilbert $A - A$ bimodule with the operations given by the restriction of the trivial Hilbert $C - C$ bimodule structure of C . Then we have $AX = X$ and $XA = X$ because both actions are automatically non-degenerate. Moreover, if we assume that X is right full, that is $X^*X = A$, then we have $X^{-k}X^l = X^{l-k}$ for $k, l \geq 0$.

2.3. *Crossed product by a Hilbert bimodule.* Crossed products of C^* -algebras by Hilbert bimodules are introduced in [2]. We summarize here their definition and principal properties.

2.3.1. *Covariant pairs.* Given a Hilbert $A - A$ bimodule X and a C^* -algebra C a covariant pair from ${}_A X_A$ to C is a pair of maps (φ, ψ) where $\varphi: A \rightarrow C$ is a $*$ -morphism and $\psi: X \rightarrow C$ a linear map satisfying

$$\begin{aligned} 1. \psi(a \cdot x) &= \varphi(a)\psi(x), & 2. \varphi(\langle x, y \rangle_L) &= \psi(x)\psi(y)^*, \\ 3. \psi(x \cdot a) &= \psi(x)\varphi(a), & 4. \varphi(\langle x, y \rangle_R) &= \psi(x)^*\psi(y), \end{aligned}$$

for all $a \in A, x, y \in X$. That is, the pair preserves the Hilbert bimodule structure considering on C the trivial Hilbert $C - C$ bimodule structure.

2.3.2. *The crossed product.* A crossed product of a C^* -algebra A by a Hilbert $A - A$ bimodule X is a C^* -algebra $A \rtimes X$ (denoted $A \rtimes_X \mathbb{Z}$ in [2]) together with a covariant pair (ι_A, ι_X) from ${}_A X_A$ to $A \rtimes X$ satisfying the following universal property: *for any covariant pair (φ, ψ) from ${}_A X_A$ to a C^* -algebra C there exists a unique $*$ -morphism $\varphi \rtimes \psi: A \rtimes X \rightarrow C$ such that $\varphi = (\varphi \rtimes \psi) \circ \iota_A$ and $\psi = (\varphi \rtimes \psi) \circ \iota_X$.*

2.3.3. *Basic properties.* The crossed product exists and is unique up to isomorphism. The maps ι_A and ι_X are injective, so that we may consider $A, X \subseteq A \rtimes X$ and the induced $*$ -morphism $\varphi \rtimes \psi$ as an extension of the covariant pair (φ, ψ) . Moreover, for any covariant pair (φ, ψ) we have that $\text{Im } \varphi \rtimes \psi = C^*(\text{Im } \varphi \cup \text{Im } \psi)$ and that $\varphi \rtimes \psi$ is injective if φ is.

3. THE MAIN THEOREM.

3.1. *Twisting Hilbert modules.* If X_B is a right Hilbert B -module, C a C^* -algebra and $\sigma: C \rightarrow B$ a $*$ -isomorphism, then we denote X_σ the right Hilbert module over C obtained by considering on the vector space X the operations

$$x \cdot_\sigma c = x \cdot \sigma(c) \quad \text{and} \quad \langle x, y \rangle^\sigma = \sigma^{-1}(\langle x, y \rangle) \quad \text{for } c \in C, x, y \in X.$$

If in addition X is a Hilbert $A - B$ bimodule then X_σ is a Hilbert $A - C$ bimodule with the original left structure. The module X is right full iff X_σ is.

3.2. The twisted sum of a cycle of Hilbert bimodules. Given Hilbert bimodules ${}_{A_i}X_{iB_i}$ for $i = 1, \dots, n$, we have that $\bigoplus X_i$ is a Hilbert $\bigoplus A_i - \bigoplus B_i$ bimodule with point-wise operations. The bimodule $\bigoplus X_i$ is right full iff X_i is for all $i = 1, \dots, n$.

Now, given a “cycle” of Hilbert bimodules ${}_{A_1}X_{1A_2}, {}_{A_2}X_{2A_3}, \dots, {}_{A_n}X_{nA_1}$ we can make $\bigoplus X_i$ into a Hilbert bimodule over $\bigoplus A_i$ twisting the right action in the previous construction with the isomorphism $\sigma: A_1 \oplus A_2 \oplus \dots \oplus A_n \rightarrow A_2 \oplus \dots \oplus A_n \oplus A_1$ given by

$$\sigma(a_1, a_2, \dots, a_n) = (a_2, \dots, a_n, a_1), \quad \text{for } a_k \in A_k.$$

Theorem 3.1. *Let ${}_{A_1}X_{1A_2}, {}_{A_2}X_{2A_3}, \dots, {}_{A_n}X_{nA_1}$ be right full Hilbert bimodules and consider their twisted sum $(X_1 \otimes \dots \otimes X_n)_\sigma$ as in 3.2. Then we have the following Morita equivalence*

$$A_1 \rtimes (X_1 \otimes \dots \otimes X_n) \sim (A_1 \oplus \dots \oplus A_n) \rtimes (X_1 \oplus \dots \oplus X_n)_\sigma.$$

Proof. Let $A = A_1 \oplus \dots \oplus A_n$, $X = (X_1 \oplus \dots \oplus X_n)_\sigma$ and $C = A \rtimes X$. We may suppose that $A_k \subseteq A \subseteq C$ and $X_k \subseteq X \subseteq C$, for $k = 1, \dots, n$, so that the module operations of each bimodule X_k and also the ones of the bimodule X are given by the operations of the C^* -algebra C , i.e., by the restriction of the trivial Hilbert $C - C$ bimodule structure of C . Note that the spaces $A, X \subseteq C$ verify the conditions in 2.2.1, then we can define X^k for $k \in \mathbb{Z}$ as done there. Moreover, as X is a Hilbert $A - A$ bimodule (hence, non-degenerate for both actions) and is right full, because each X_k is, we have that $A, X \subseteq C$ verify the conditions of 2.2.3.

We extend the families $\{A_k\}_{k=1}^n$ and $\{X_k\}_{k=1}^n$ to families $\{A_k\}_{k \in \mathbb{Z}}$ and $\{X_k\}_{k \in \mathbb{Z}}$ letting $A_k = A_l$ and $X_k = X_l$ if $k = l \bmod n$. For all $k \in \mathbb{Z}$ we have

$$A_k X_k = X_k = X_k A_{k+1}, \quad X_k X_k^* \subseteq A_k, \quad \text{and} \quad X_k^* X_k = A_{k+1},$$

because each X_k is a Hilbert $A_k - A_{k+1}$ bimodule (hence non-degenerate for both actions) and right full. We also have

$$A_k A_l = A_k X_l = X_k A_{l+1} = 0 \quad \text{for } k, l \in \mathbb{Z}, \quad k \neq l \bmod n,$$

therefore, as $A = \sum_{k=1}^n A_k$ and $X = \sum_{k=1}^n X_k$,

$$A_k = A_k A = A A_k, \quad X_k = A_k X = X A_{k+1} \quad \text{for } k \in \mathbb{Z},$$

and then

$$A_k X^l = X^l A_{k+l} \quad \text{for } k, l \in \mathbb{Z}.$$

In particular, for $k \in \mathbb{Z}$ we have that $A_k X^n = X^n A_k$ so that the pairs $A_k, A_k X^n$ satisfy the conditions of 2.2.1 and 2.2.3. Following the notation in 2.2.1 we define the C^* -subalgebra

$$B = A_1[A_1 X^n] = C^*(A_1 \cup A_1 X^n) \subseteq C,$$

and the closed subspace

$$Z = B X^0 \oplus B X \oplus \dots \oplus B X^{n-1} \subseteq M_{1 \times n}(C),$$

where $M_{1 \times n}(C)$ is considered as a Hilbert $C - M_n(C)$ bimodule with the usual matrix operations. To prove the theorem it is enough to show that $ZZ^* \subseteq C$ and

$Z^*Z \subseteq M_n(C)$ are C^* -subalgebras, that Z is an equivalence ZZ^*-Z^*Z bimodule with the restricted (matrix) operations and that we have isomorphisms $ZZ^* \cong A_1 \rtimes (X_1 \otimes \cdots \otimes X_n)$ and $Z^*Z \cong C$.

Let us consider the issues concerning the left side first. Notice that the equalities $AA_1 = A_1 = A_1A$ and $A(A_1X^n) = A_1X^n = (A_1X^n)A$ implies, by 2.2.2, that $AB = B = BA$, because $B = A_1[A_1X^n]$ by definition. Then we can calculate

$$ZZ^* = \sum_{k=0}^{n-1} (BX^k)(BX^k)^* = B \left(\sum_{k=0}^{n-1} X^k X^{-k} \right) B = BAB = B,$$

where we use that $\sum_{k=0}^{n-1} X^k X^{-k} = A$, which is a consequence of the relations $X^k X^{-k} \subseteq A$ and $X^0 = A$, and that $AB = B$. Besides, from the definition of Z it is apparent that Z is B -invariant for the left action. Then we have shown that Z is a full left Hilbert B -module. Finally, to see that $B \cong A_1 \rtimes (X_1 \otimes \cdots \otimes X_n)$, consider the covariant pair (φ, ψ) given by

$$\varphi: A_1 \rightarrow C, \quad \varphi(a) = a \quad \text{for } a \in A_1,$$

$$\psi: X_1 \otimes \cdots \otimes X_n \rightarrow C, \quad \psi(x_1 \otimes \cdots \otimes x_n) = x_1 \cdots x_n \quad \text{for } x_k \in X_k.$$

By the universal property of the crossed product (2.3.2) this covariant pair extends to a $*$ -morphism $\varphi \rtimes \psi: A_1 \rtimes (X_1 \otimes \cdots \otimes X_n) \rightarrow C$, which is injective because φ is (2.3.3). Moreover, $\text{Im } \varphi \rtimes \psi = C^*(\text{Im } \varphi \cup \text{Im } \psi) = B$ because $\text{Im } \varphi = A_1$, $\text{Im } \psi = X_1 \cdots X_n = A_1 X \cdots A_n X = A_1 X^n$ and $B = C^*(A_1 \cup A_1 X^n)$ by definition. Hence we have the desired isomorphism.

Now, turning to the right side, note that Z^*Z is clearly a closed self-adjoint subspace of $M_n(C)$. From the equalities $ZZ^* = B$ and $BZ = Z$ we also deduce that $(Z^*Z)(Z^*Z) = Z^*BZ = Z^*Z$, so that Z^*Z is a C^* -subalgebra of $M_n(C)$ and Z is a full right Hilbert Z^*Z -module.

To show that $Z^*Z \cong C$, consider the pair of maps $\varphi: A \rightarrow M_n(C)$ and $\psi: X \rightarrow M_n(C)$ given by

$$\varphi(a_1, \dots, a_n) = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix}, \quad \psi(x_1, \dots, x_n) = \begin{bmatrix} 0 & x_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1} \\ x_n & \cdots & 0 & 0 \end{bmatrix},$$

for $(a_1, \dots, a_n) \in A = A_1 \oplus \cdots \oplus A_n$ and $(x_1, \dots, x_n) \in X = (X_1 \oplus \cdots \oplus X_n)_\sigma$.

The following calculations shows that this pair is a covariant pair. For every $a_k \in A_k$, $x_k, y_k \in X_k$, $k = 1, \dots, n$ we have

$$\begin{aligned} \psi((a_1, \dots, a_n) \cdot (x_1, \dots, x_n)) &= \psi(a_1 \cdot x_1, \dots, a_n \cdot x_n) \\ &= \begin{bmatrix} 0 & a_1 \cdot x_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1} \cdot x_{n-1} \\ a_n \cdot x_n & \cdots & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix} \begin{bmatrix} 0 & x_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1} \\ x_n & \cdots & 0 & 0 \end{bmatrix} \\ &= \varphi(a_1, \dots, a_n) \psi(x_1, \dots, x_n); \end{aligned}$$

$$\begin{aligned}
\varphi(\langle(x_1, \dots, x_n), (y_1, \dots, y_n)\rangle_L) &= \varphi(\langle x_1, y_1 \rangle_L, \dots, \langle x_n, y_n \rangle_L) \\
&= \begin{bmatrix} \langle x_1, y_1 \rangle_L & 0 & \cdots & 0 \\ 0 & \langle x_2, y_2 \rangle_L & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \langle x_n, y_n \rangle_L \end{bmatrix} = \begin{bmatrix} 0 & x_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1} \\ x_n & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & y_n^* \\ y_1^* & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & y_{n-1}^* & 0 \end{bmatrix} \\
&= \psi(x_1, \dots, x_n) \psi(y_1, \dots, y_n)^*;
\end{aligned}$$

$$\begin{aligned}
\psi((x_1, \dots, x_n) \cdot_\sigma (a_1, \dots, a_n)) &= \psi((x_1, \dots, x_n) \cdot \sigma(a_1, \dots, a_n)) \\
&= \psi((x_1, \dots, x_n) \cdot (a_2, \dots, a_n, a_1)) = \psi(x_1 \cdot a_2, \dots, x_{n-1} \cdot a_n, x_n \cdot a_1) \\
&= \begin{bmatrix} 0 & x_1 \cdot a_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1} \cdot a_n \\ x_n \cdot a_1 & \cdots & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1} \\ x_n & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix} \\
&= \psi(x_1, \dots, x_n) \varphi(a_1, \dots, a_n);
\end{aligned}$$

$$\begin{aligned}
\varphi(\langle(x_1, \dots, x_n), (y_1, \dots, y_n)\rangle_R^\sigma) &= \varphi(\sigma^{-1}(\langle(x_1, \dots, x_n), (y_1, \dots, y_n)\rangle_R)) \\
&= \varphi(\sigma^{-1}(\langle x_1, y_1 \rangle_R, \dots, \langle x_n, y_n \rangle_R)) = \varphi(\langle x_n, y_n \rangle_R, \langle x_1, y_1 \rangle_R, \dots, \langle x_{n-1}, y_{n-1} \rangle_R) \\
&= \begin{bmatrix} \langle x_n, y_n \rangle_R & 0 & \cdots & 0 \\ 0 & \langle x_1, y_1 \rangle_R & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \langle x_{n-1}, y_{n-1} \rangle_R \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & x_n^* \\ x_1^* & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & x_{n-1}^* & 0 \end{bmatrix} \begin{bmatrix} 0 & y_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_{n-1} \\ y_n & \cdots & 0 & 0 \end{bmatrix} \\
&= \psi(x_1, \dots, x_n)^* \psi(y_1, \dots, y_n).
\end{aligned}$$

By the universal property of the crossed product the covariant pair (φ, ψ) extends to a $*$ -morphism $\varphi \rtimes \psi: A \rtimes X \rightarrow M_n(C)$, which is injective because φ is. Then, to end the proof it suffices to show that $\text{Im } \varphi \rtimes \psi = Z^*Z$.

We calculate $Z^*Z \subseteq M_n(C)$ adopting the following matrix notation

$$Z^*Z = [E_{ij}]_{i,j=1}^n \quad \text{where} \quad E_{ij} = (BX^{i-1})^*(BX^{j-1}) \subseteq C, \quad \text{for } i, j = 1, \dots, n.$$

Simplifying the expressions of the E_{ij} 's we get

$$E_{ij} = (BX^{i-1})^*(BX^{j-1}) = X^{1-i}BX^{j-1}, \quad \text{for } i, j = 1, \dots, n.$$

Note that

$$E_{ii} = X^{1-i}BX^{i-1} \supseteq X^{1-i}A_1X^{i-1} = X^{1-i}X^{i-1}A_i = A_i \quad \text{for } i = 1, \dots, n,$$

$$E_{ii+1} = E_{ii}X \supseteq A_iX = X_i \quad \text{for } i = 1, \dots, n-1 \quad \text{and}$$

$$E_{n1} = X^{1-n}BX^0 \supseteq X^{1-n}(A_1X^n)A = A_nX^{1-n}X^n = A_nX = X_n.$$

Then we see that $\text{Im } \varphi \subseteq Z^*Z$ and $\text{Im } \psi \subseteq Z^*Z$. As $\text{Im } \varphi \rtimes \psi = C^*(\text{Im } \varphi \cup \text{Im } \psi)$ we conclude that $\text{Im } \varphi \rtimes \psi \subseteq Z^*Z$.

To prove the reverse inclusion denote $D = \text{Im } \varphi \rtimes \psi = C^*(\text{Im } \varphi \cup \text{Im } \psi)$,

$$\text{Im } \varphi = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_n \end{bmatrix} \subseteq D \quad \text{and} \quad \tilde{X} = \text{Im } \psi = \begin{bmatrix} 0 & X_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & X_{n-1} \\ X_n & \cdots & 0 & 0 \end{bmatrix} \subseteq D.$$

Note that

$$\begin{aligned} & \begin{bmatrix} 0 & X_1 & \cdots & 0 \\ 0 & 0 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & X_2 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & X_{n-1} \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ X_n & \cdots & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} X_1 \cdots X_n & 0 & \cdots & 0 \\ 0 & 0 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 X^n & 0 & \cdots & 0 \\ 0 & 0 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \subseteq D. \end{aligned}$$

Then, with $\begin{bmatrix} A_1 & 0_{1 \times (n-1)} \\ 0_{(n-1) \times 1} & 0_{(n-1) \times (n-1)} \end{bmatrix} \subseteq D$ and $\begin{bmatrix} A_1 X^n & 0_{1 \times (n-1)} \\ 0_{(n-1) \times 1} & 0_{(n-1) \times (n-1)} \end{bmatrix} \subseteq D$ we can generate $\tilde{E}_{11} = \begin{bmatrix} E_{11} & 0_{1 \times (n-1)} \\ 0_{(n-1) \times 1} & 0_{(n-1) \times (n-1)} \end{bmatrix} \subseteq D$, because $E_{11} = B = A_1[A_1 X^n]$. Now, for $k, l = 0, \dots, n-1$ we have that the product $\tilde{X}^{-k} \tilde{E}_{11} \tilde{X}^l \subseteq D$ is the space that, with the matrix notation, has $X_k^* \cdots X_1^* B X_1 \cdots X_l = X^{-k} B X^l = E_{1+k, 1+l}$ at the $(l+1, k+1)$ -entry and 0 elsewhere. As $(k+1, l+1)$ ranges over all entries when $k, l = 0, \dots, n-1$, we conclude that $Z^* Z = [E_{ij}]_{i,j=1}^n \subseteq D = \text{Im } \varphi \rtimes \psi$ as desired. \square

Remark 3.2. For the especial case of Theorem 3.1 in which $A_i = A$ and $X_i = X$ for $i = 1, \dots, n$, we obtain $A^n \rtimes X_\sigma^n \sim A \rtimes X^{\otimes n}$. As pointed out in the introduction, this can be viewed as a generalization to the C^* -bimodule context of Green's theorem [5, Theorem 4.22] for the case $G = \mathbb{Z}$ and $H = n\mathbb{Z}$. That is, if $\alpha: A \rightarrow A$ is a $*$ -automorphism, taking X as the trivial Hilbert bimodule ${}_A A_A$ twisted by α , the equivalence $A^n \rtimes X_\sigma^n \sim A \rtimes X^{\otimes n}$ becomes $C_0(G/H, A) \rtimes G \sim A \rtimes_{\alpha|_H} H$ where α also denotes the action of \mathbb{Z} generated by the automorphism.

Corollary 3.3. *Let ${}_A X_B$ and ${}_B Y_A$ be full right Hilbert bimodules. Then*

$$A \rtimes (X \otimes Y) \sim B \rtimes (Y \otimes X).$$

Proof. The twisted sums $(X \oplus Y)_\sigma$ and $(Y \oplus X)_\sigma$ are isomorphic bimodules, the pair (φ, ψ) where $\varphi: A \oplus B \rightarrow B \oplus A$, $\varphi(a, b) = (b, a)$, and $\psi: X \oplus Y \rightarrow Y \oplus X$, $\psi(x, y) = (y, x)$, being an isomorphism. Then, the corresponding crossed products are isomorphic as well. Therefore

$$A \rtimes (X \otimes Y) \sim (A \oplus B) \rtimes (X \oplus Y)_\sigma \cong (B \oplus A) \rtimes (Y \oplus X)_\sigma \sim B \rtimes (Y \otimes X),$$

where we applied twice Theorem 3.1 for $n = 2$. \square

Corollary 3.4. ([2, Theorem 4.1]) *Let ${}_AX_A$ and ${}_BY_B$ be full right Hilbert bimodules and ${}_AM_B$ an equivalence bimodule such that $X \otimes M \cong M \otimes Y$. Then*

$$A \rtimes X \sim B \rtimes Y.$$

Proof. As M is an equivalence we have $A \cong M \otimes M^*$, where A is considered as the trivial Hilbert $A - A$ bimodule and M^* denotes the conjugated bimodule of M . Then $X \cong A \otimes X \cong M \otimes M^* \otimes X$. Besides, $M^* \otimes X \otimes M \cong Y$ by hypothesis. Then, as all these isomorphisms give isomorphic crossed products, we have

$$A \rtimes X \cong A \rtimes (A \otimes X) \cong A \rtimes (M \otimes M^* \otimes X) \sim B \rtimes (M^* \otimes X \otimes M) \cong B \rtimes Y,$$

where we applied Corollary 3.3 to commute M and $M^* \otimes X$. \square

Remark 3.5. In [1] the augmented Cuntz-Pimsner C^* -algebra $\tilde{\mathcal{O}}_X$ associated to an $A - A$ correspondence X (see [4]) is described as a crossed product $A_\infty \rtimes X_\infty$, where X_∞ is a Hilbert $A_\infty - A_\infty$ bimodule constructed out of the $A - A$ correspondence X . Then, combining this description with [2, Theorem 4.1] (Corollary 3.4 here) it is shown an analogue of this theorem in the context of augmented Cuntz-Pimsner C^* -algebras ([1, Theorem 4.7]).

We believe that using similar techniques to those of [1], it is possible to obtain versions of Theorem 3.1 and Corollary 3.3 for augmented Cuntz-Pimsner algebras. For example, the corresponding version of the Corollary 3.3 should establish that $\tilde{\mathcal{O}}_{X \otimes Y} \sim \tilde{\mathcal{O}}_{Y \otimes X}$ for full correspondences ${}_AX_B$ and ${}_BY_A$.

Acknowledgement. The author wishes to thank his friend Janine Bachrachas for her help editing this article.

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